

AN EXTENSION OF NUNOKAWA LEMMA AND ITS EXAMPLE

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ABSTRACT. For analytic functions $p(z)$ in the open unit disk \mathbb{U} with $p(0) = 1$, Nunokawa has given a result which called Nunokawa lemma (Proc. Japan Acad., Ser. A **68** (1992)). By studying Nunokawa lemma, we obtain this expansion. In this paper, we introduce this result and its example.

1. INTRODUCTION

Let \mathbb{U} be defined by the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

The basic tool in proving our results is the following lemma due to Miller and Mocanu [1] (also [2]).

Lemma 1. *Let the function $w(z)$ be analytic in \mathbb{U} with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathbb{U}$, then there exists a real number $m \geq 1$ such that*

$$\frac{z_0 w'(z_0)}{w(z_0)} = m.$$

2. MAIN RESULT

Applying Lemma 1, we derive the following result.

Theorem 1. *Let $p(z)$ be analytic in \mathbb{U} with $p(0) = 1$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$\operatorname{Re}(p(z)) > \alpha \quad \text{for} \quad |z| < |z_0|$$

and

$$p(z_0) = \alpha + \beta i$$

for some real α and β , $0 \leq \alpha < 1$ and $\beta \neq 0$.

Then we have

$$\operatorname{Re} \left(\frac{z_0 p'(z_0)}{p(z_0)} \right) = -\frac{\alpha \beta k}{\alpha^2 + \beta^2} \leq 0$$

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and

$$\operatorname{Im} \left(\frac{z_0 p'(z_0)}{p(z_0)} \right) = \frac{\beta^2 k}{\alpha^2 + \beta^2}$$

where

$$k \geq \frac{1}{2} \left(\frac{\beta}{1-\alpha} + \frac{1-\alpha}{\beta} \right) \geq 1 \quad (\beta > 0)$$

and

$$k \leq \frac{1}{2} \left(\frac{\beta}{1-\alpha} + \frac{1-\alpha}{\beta} \right) \leq -1 \quad (\beta < 0).$$

Proof. Let us define the function $q(z)$ by

$$q(z) = \frac{p(z) - \alpha}{1 - \alpha} \quad (z \in \mathbb{U}).$$

Clearly, $q(z)$ is analytic in \mathbb{U} with $q(0) = 1$ and $q(z_0) = \frac{\beta}{1-\alpha}i$ for a point z_0 such that $p(z_0) = \alpha + \beta i$.

Also, let us put

$$w(z) = \frac{1 - q(z)}{1 + q(z)} \quad (z \in \mathbb{U}).$$

Then, we have that $w(z)$ is analytic in $|z| < |z_0|$, $w(0) = 0$, $|w(z)| < 1$ for $|z| < |z_0|$ and

$$|w(z_0)| = \left| \frac{(1-\alpha)^2 - \beta^2 - 2(1-\alpha)\beta i}{(1-\alpha)^2 + \beta^2} \right| = 1.$$

From Lemma 1, we obtain

$$\frac{z_0 w'(z_0)}{w(z_0)} = \frac{-2z_0 q'(z_0)}{1 - \{q(z_0)\}^2} = \frac{-2z_0 q'(z_0)}{1 + \left(\frac{\beta}{1-\alpha} \right)^2} = m \geq 1.$$

This shows that

$$-z_0 q'(z_0) \geq \frac{1}{2} \left(1 + \left(\frac{\beta}{1-\alpha} \right)^2 \right)$$

and $z_0 q'(z_0)$ is a negative real number.

From the fact that $z_0 q'(z_0)$ is a real number and $q(z_0)$ is a pure imaginary number, we can put

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik$$

where k is a real number.

For the case $\beta > 0$, we have

$$\begin{aligned}
k &= \operatorname{Im} \left(\frac{z_0 q'(z_0)}{q(z_0)} \right) \\
&= \operatorname{Im} \left(-z_0 q'(z_0) \frac{1-\alpha}{\beta} i \right) \\
&\geq \frac{1}{2} \left(1 + \left(\frac{\beta}{1-\alpha} \right)^2 \right) \frac{1-\alpha}{\beta} \\
&= \frac{1}{2} \left(\frac{\beta}{1-\alpha} + \frac{1-\alpha}{\beta} \right) \geq 1
\end{aligned}$$

and for the case $\beta < 0$, we get

$$\begin{aligned}
k &= \operatorname{Im} \left(\frac{z_0 q'(z_0)}{q(z_0)} \right) \\
&= \operatorname{Im} \left(-z_0 q'(z_0) \frac{1-\alpha}{\beta} i \right) \\
&\leq \frac{1}{2} \left(1 + \left(\frac{\beta}{1-\alpha} \right)^2 \right) \frac{1-\alpha}{\beta} \\
&= \frac{1}{2} \left(\frac{\beta}{1-\alpha} + \frac{1-\alpha}{\beta} \right) \leq -1.
\end{aligned}$$

On the other hand, let us consider

$$\frac{z_0 q'(z_0)}{q(z_0)} = \frac{z_0 p'(z_0)}{p(z_0) - \alpha} = ik,$$

then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{p(z_0) - \alpha}{p(z_0)} ik = -\frac{\alpha \beta k}{\alpha^2 + \beta^2} + \frac{\beta^2 k}{\alpha^2 + \beta^2} i.$$

This completes our proof. \square

Putting $\alpha = 0$ in Theorem 1, we have Corollary 1 [3].

Corollary 1. *Let $p(z)$ be analytic in \mathbb{U} with $p(0) = 1$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$\operatorname{Re}(p(z)) > 0 \quad \text{for} \quad |z| < |z_0|,$$

$\operatorname{Re}(p(z_0)) = 0$ and $p(z_0) \neq 0$.

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik$$

where k is a real and $k \geq 1$ for $\operatorname{Im}(p(z_0)) > 0$ and $k \leq -1$ for $\operatorname{Im}(p(z_0)) < 0$.

3. EXAMPLE OF THE THEOREM

Example 1. We consider the function $p(z)$ given by

$$p(z) = 1 + (1 - \alpha)(2z + z^2) \quad (z \in \mathbb{U})$$

for some real $0 \leq \alpha < 1$.

Then, $p(z)$ is analytic in \mathbb{U} with $p(0) = 1$.

Putting $z_0 = -\frac{1}{2} \pm \frac{1}{2}i$, it follows that

$$\operatorname{Re}(p(z)) > \alpha \quad \text{for } |z| < |z_0| = \frac{1}{\sqrt{2}}$$

and $\operatorname{Re}(p(z_0)) = \alpha$. For the case $z_0 = -\frac{1}{2} + \frac{1}{2}i$, we have

$$p(z_0) = \alpha + \frac{1 - \alpha}{2}i.$$

Putting $\beta = \frac{1 - \alpha}{2}$, we obtain

$$\frac{z_0 p'(z_0)}{p(z_0)} = -\frac{4\alpha(1 - \alpha)}{4\alpha^2 + (1 - \alpha)^2} + \frac{2(1 - \alpha)^2}{4\alpha^2 + (1 - \alpha)^2}i = -\frac{\alpha\beta k}{\alpha^2 + \beta^2} + \frac{\beta^2 k}{\alpha^2 + \beta^2}i$$

where

$$k = 2 \geq \frac{5}{4} = \frac{1}{2} \left(\frac{\beta}{1 - \alpha} + \frac{1 - \alpha}{\beta} \right).$$

For the case $z_0 = -\frac{1}{2} - \frac{1}{2}i$, we have

$$p(z_0) = \alpha - \frac{1 - \alpha}{2}i.$$

Putting $\beta = -\frac{1 - \alpha}{2}$, we obtain also

$$\frac{z_0 p'(z_0)}{p(z_0)} = -\frac{4\alpha(1 - \alpha)}{4\alpha^2 + (1 - \alpha)^2} - \frac{2(1 - \alpha)^2}{4\alpha^2 + (1 - \alpha)^2}i = -\frac{\alpha\beta k}{\alpha^2 + \beta^2} + \frac{\beta^2 k}{\alpha^2 + \beta^2}i$$

where

$$k = -2 \leq -\frac{5}{4} = \frac{1}{2} \left(\frac{\beta}{1 - \alpha} + \frac{1 - \alpha}{\beta} \right).$$

The function $p(z)$ satisfies Theorem 1.

Especially, the function

$$p(z) = 1 + z + \frac{1}{2}z^2 \quad (z \in \mathbb{U})$$

is one of the example of Theorem Theorem 1. In fact, when we choice a point z_0 such that

$$z_0 = -\frac{1}{2} \pm \frac{1}{2}i$$

for $|z_0| = \frac{1}{\sqrt{2}}$, the function $p(z)$ satisfies that $\operatorname{Re}(p(z_0)) > \frac{1}{2}$ for $|z| < |z_0|$ and $\operatorname{Re}(p(z_0)) = \frac{1}{2}$.

For $z_0 = -\frac{1}{2} + \frac{1}{2}i$, we have

$$p(z_0) = \frac{1}{2} + \frac{1}{4}i$$

and

$$\frac{z_0 p'(z_0)}{p(z_0)} = -\frac{4}{5} + \frac{2}{5}i = -\frac{2}{5}k + \frac{1}{5}ki$$

with $k = 2 \geq \frac{5}{4}$. Furthermore, for $z_0 = -\frac{1}{2} - \frac{1}{2}i$, we get

$$p(z_0) = \frac{1}{2} - \frac{1}{4}i$$

and

$$\frac{z_0 p'(z_0)}{p(z_0)} = -\frac{4}{5} - \frac{2}{5}i = \frac{2}{5}k + \frac{1}{5}ki$$

with $k = -2 \leq -\frac{5}{4}$.

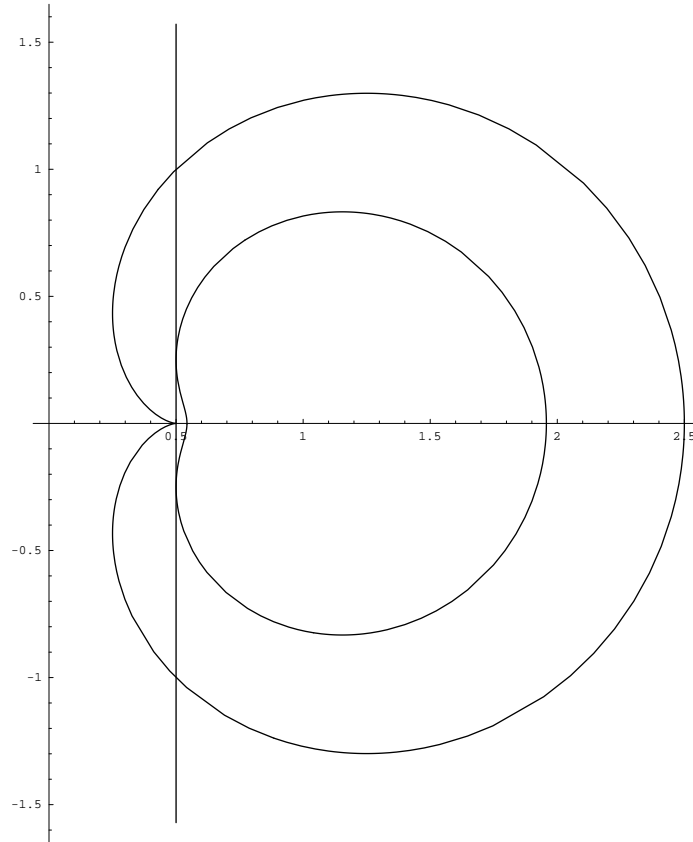


FIGURE 1. $p(z) = 1 + z + \frac{1}{2}z^2$ in $|z| = 1$ and $|z| = \frac{1}{\sqrt{2}}$

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